The Complexity of the Nonsplitting Family and the Splitting Ideal

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The question

- Sabok and Zapletal (2011) introduced the Borel nonsplitting family SPL. Its exact Borel complexity remained to be determined.
- The generated ideal $\mathcal{S}pl$ is analytic. They asked whether it is Borel. Cieślak (2026) proved that it is not Π^0_4.

Our results

SPL is Σ^0_4-complete, and $\mathcal{S}pl$ is Σ^1_1-complete.

Outline

1 Definitions and main results

2 The nonsplitting family

3 The splitting ideal

Definitions and main results

Nonsplitting family and splitting ideal

For $c \in [\omega]^\omega$, let

$$S(c) = \big\{ t \in 2^{<\omega} : t \restriction (c \cap |t|) \text{ is constant} \big\}.$$

For example, let $c = 2\mathbb{N} = \{0, 2, 4, \cdots\}$. $t_0 = (1, 1, 1, 0, 1, 0) \in S(c)$, $t_1 = (0, 1, 0, 0, 0) \in S(c)$, whereas $t_2 = (1, 1, 0, 0, 1, 0) \notin S(c)$.

Definition

Nonsplitting family

$$\mathrm{SPL} = \big\{ a \subseteq 2^{<\omega} : \exists c \in [\omega]^\omega \ \big(a \subseteq S(c) \big) \big\}.$$

Splitting ideal

$$\mathcal{S}pl = \left\{ a \subseteq 2^{<\omega} : \exists k < \omega \ \exists c_0, \ldots, c_{k-1} \in [\omega]^\omega \ \left(a \subseteq \bigcup_{i<k} S(c_i) \right) \right\}.$$

We identify subsets of $2^{<\omega}$ with their characteristic functions. Thus $\mathrm{SPL}, \mathcal{S}pl \subseteq 2^{2^{<\omega}} \cong 2^\omega$, with the product topology.

Main results

Theorem A

SPL is Σ^0_4-complete.

Theorem B

$\mathcal{S}pl$ is Σ^1_1-complete.

The nonsplitting family

An equivalent characterization

Suppose $a \subseteq S(c)$ for some $c \in [\omega]^\omega$ and let $n = \min c$. Every $k \in c$ satisfies

$$\forall t \in a \left(k < |t| \implies t(k) = t(n)\right).$$

Collect them all,

$$E_n(a) = \left\{k \geq n : \forall t \in a \left(k < |t| \implies t(k) = t(n)\right)\right\}.$$

Claim

$$a \in \mathrm{SPL} \iff \exists n \in \omega,\ E_n(a) \text{ is infinite } \Big(\iff \exists n \exists^\infty k, k \in E_n(a)\Big).$$

Here, $\exists^\infty k$ means $\forall M, \exists k > M$.

- If $a \subseteq S(c)$, take $n = \min c$; then $c \subseteq E_n(a)$.
- If $E_n(a)$ is infinite, take $c = E_n(a)$.

SPL is Σ^0_4

The condition $k \in E_n(a)$ is closed. To translate $k \in E_n(a)$, define, for $n \leq k$,

$$D_{n,k} = \left\{ t \in 2^{<\omega} : |t| > k \text{ and } t(n) \neq t(k) \right\}.$$

Then $k \in E_n(a) \iff a \cap D_{n,k} = \varnothing \iff \forall t \in D_{n,k}, t \notin a$. For a fixed t, $\{a \in 2^{2^{<\omega}} : t \notin a\}$ is clopen.

Hence

$$a \in \mathrm{SPL} \iff \exists n \, \exists^{\infty} k > n \, \forall t \in D_{n,k}, \ t \notin a.$$

Equivalently,

$$\mathrm{SPL} = \bigcup_{n \in \omega} \bigcap_{M \in \omega} \bigcup_{k > \max\{M,n\}} \underbrace{\bigcap_{t \in D_{n,k}} \{a \subseteq 2^{<\omega} : t \notin a\}}_{\text{closed}}.$$

Thus $\mathrm{SPL} \in \Sigma^0_4$.

Choose a known Σ^0_4-complete set

To prove Σ^0_4-hardness, recall

$$a \in \mathrm{SPL} \iff \exists n\, \exists^\infty k > n\, \forall t \in D_{n,k}\, (t \notin a).$$

So we choose the following Σ^0_4-complete set with a similar quantifier structure:

$$S_4 = \big\{x \in 2^{\omega\times\omega\times\omega} : \exists n \quad \exists^\infty k \quad \forall \ell \quad x(n,k,\ell) = 1\big\}.$$

$$\mathrm{SPL} = \{a \subseteq 2^{<\omega} : \exists n\ \exists^\infty k > n\ \forall t \in D_{n,k}\ \ t \notin a.\}$$

We will design a continuous f such that

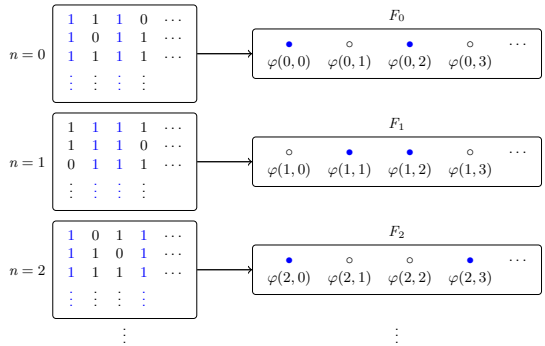
$$x \in S_4 \iff f(x) \in \mathrm{SPL}.$$

Designing f : $x \in S_4 \implies f(x) \in \mathrm{SPL}$

Suppose $x \in S_4$.

Key point: find the witness c. Naturally, let c be the collection of infinitely many k. So we need to fix a bijection $\varphi : \omega \times \omega \to \omega$, and put

$$F_n = \{\varphi(n,k) : k \in \omega\} \subseteq \omega.$$



If the n-th slice has infinitely many all-one columns, collect all the blue points in F_n to form c.

Designing f: $x \in S_4 \implies f(x) \in \mathrm{SPL}$

Let $x \in S_4$. A column $(n, k, \cdot)$ is called bad for x if $x(n, k, \ell) = 0$ for some ℓ.
For a bad column $(n, k, \cdot)$, let $p = \varphi(n, k)$. We put a sequence $q_{n,k,\ell}$ into $f(x)$ to prevent $p \in c$.

$$q_{n,k,\ell}(i) = \begin{cases} 1, & i = p, \\ 0, & i \neq p, \end{cases} \qquad i < \max\{p, \ell\} + 1.$$

Let Q_x be the collection of $q_{n,k,l}$ for the bad columns $(n, k, \cdot)$.

$$Q_x = \{q_{n,k,\ell} : x(n, k, \ell) = 0\}.$$

Here the length of q involves ℓ to ensure f is continuous.
Thus a bad coordinate $p = \varphi(n, k)$ cannot belong to a witness c unless it is its least element.

Designing f: $x \in S_4 \Longleftarrow f(x) \in \mathrm{SPL}$

Suppose $Q_x \in \mathrm{SPL}$. Then there is an infinite $c \subseteq \omega$ with $Q_x \subseteq S(c)$.
Naturally, we want c to provide infinitely many k. But the sequences in Q_x do not prevent a witness from mixing coordinates from different sets F_n. So we put some $h_{n,k}$ into $f(x)$ to prevent that, for every n, k,

$$h_{n,k}(i) = \begin{cases} 0, & i \in F_n, \\ 1, & i \notin F_n, \end{cases} \qquad i < \varphi(n,k) + 1.$$

Put $H = \{h_{n,k} : n, k \in \omega\}$. Here the length of $h_{n,k}$ does not need to involve ℓ because H is fixed and does not depend on x. So adding H into $f(x)$ will not cause problems for continuity of f.
Thus any infinite c with $H \subseteq S(c)$ must lie in a single F_n.

Finally, let $f(x) = Q_x \cup H$.

Proof sketch: $x \in S_4 \iff f(x) \in \mathrm{SPL}$

($\Rightarrow$) If $x \in S_4$, choose n with infinitely many all-one columns and put

$$c = \{\varphi(n,k) : \forall \ell\ x(n,k,\ell) = 1\}.$$

Then c is infinite. Every $q \in Q_x$ is constantly 0 on $c \cap |q|$, and every $h \in H$ is constant on $c \cap |h|$. Hence $f(x) \subseteq S(c)$, so $f(x) \in \mathrm{SPL}$.

($\Leftarrow$) Suppose $f(x) \subseteq S(c)$ for some infinite c. By H, we have $c \subseteq F_n$ for some n.
If $p = \varphi(n,k) \in c \setminus \{\min c\}$ and $x(n,k,\ell) = 0$, then $q_{n,k,\ell} \in Q_x$ satisfies

$$q_{n,k,\ell}(\min c) = 0 \neq 1 = q_{n,k,\ell}(p),$$

a contradiction. Thus infinitely many k satisfy $\forall \ell\ x(n,k,\ell) = 1$, so $x \in S_4$.

Since f is continuous, $S_4 \leq_W \mathrm{SPL}$. Together with $\mathrm{SPL} \in \Sigma^0_4$, this proves:

Theorem A

SPL is Σ^0_4-complete.

The splitting ideal

$\mathcal{S}pl$ is analytic

Membership in $\mathcal{S}pl$ asks for finitely many infinite witnesses:

$$a \in \mathcal{S}pl \iff \exists k < \omega\ \exists c_0, \dots, c_{k-1} \in [\omega]^\omega\ \forall t \in a\ \exists i < k\ (t \in S(c_i)).$$

For each fixed k, the relation

$$R_k(a, c_0, \dots, c_{k-1}) \iff a \subseteq \bigcup_{i<k} S(c_i)$$

is closed in $2^{2^{<\omega}} \times ([\omega]^\omega)^k$.

Fact

Projecting over $c_0, \dots, c_{k-1}$ and taking the countable union over k gives $\mathcal{S}pl \in \Sigma^1_1$.

The remaining task is Σ^1_1-hardness.

Reduction from IF to $\mathcal{S}pl$

Let Tr be the space of trees on ω, and let

$$\mathrm{IF} = \{ T \in \mathrm{Tr} : T \text{ has an infinite branch} \}.$$

IF is Σ^1_1-complete. We will construct a continuous reduction g such that

$$T \in \mathrm{IF} \iff g(T) \in \mathcal{S}pl.$$

Designing g: $T \in \mathrm{IF} \implies g(T) \in \mathcal{S}pl$

Suppose T has an infinite branch $x \in \omega^\omega$. The key point: find the witnesses $c_0, c_1, \cdots, c_{k-1}$.

The first attempt

A natural idea is to use an infinite branch to obtain a witness. Fix a bijection $\psi : \omega \to \omega^{<\omega}$. For a branch $x \in [T]$, put

$$D = \{x \restriction n : n \in \omega\},$$

$$c = \psi^{-1}[D] = \{\psi^{-1}(x \restriction n) : n \in \omega\}.$$

However, our desired continuous reduction cannot have a single witness for every $T \in \mathrm{IF}$. Together with the reverse implication, this would give $\mathrm{IF} \leq_W \mathrm{SPL}$, which is impossible because SPL is Borel.

We therefore split c into two infinite sets c_0, c_1 and aim for

$$g(T) \subseteq S(c_0) \cup S(c_1).$$

Designing g

A sufficient condition for the two-set cover is:

$$\forall t \in g(T) \qquad |t^{-1}(1) \cap c| \leq 1.$$

In words, each t has at most one 1 on the branch coordinates.
That 1 can lie in at most one of c_0, c_1. On the other part, t is constantly 0 wherever defined. Hence

$$t \in S(c_0) \cup S(c_1).$$

To handle every possible branch, require at most one 1 on the coordinates of *every chain* in T. Put

$$E_t = \psi[t^{-1}(1)].$$

A set meets every chain in at most one node exactly when it is an antichain. This leads to:

The reduction

$$g(T) = \{t \in 2^{<\omega} : E_t \cap T \text{ is an antichain}\}.$$

Proof sketch: $T \in \mathrm{IF} \implies g(T) \in \mathcal{S}pl$

Suppose $x \in [T]$. Let

$$c_0 = \{\psi^{-1}(x \upharpoonright 2m) : m \in \omega\},$$

$$c_1 = \{\psi^{-1}(x \upharpoonright (2m+1)) : m \in \omega\}.$$

Claim

$g(T) \subseteq S(c_0) \cup S(c_1)$, so $g(T) \in \mathcal{S}pl$.

For $t \in g(T)$, the antichain $E_t \cap T$ meets the chain $\{x \upharpoonright m : m \in \omega\}$ in at most one node. WLOG, suppose $E_t \cap T \cap \{x \upharpoonright m : m \in \omega\} = \{s\}$ where $\psi^{-1}(s) \in c_0$, thus t is constantly 0 on $c_1 \cap |t|$.

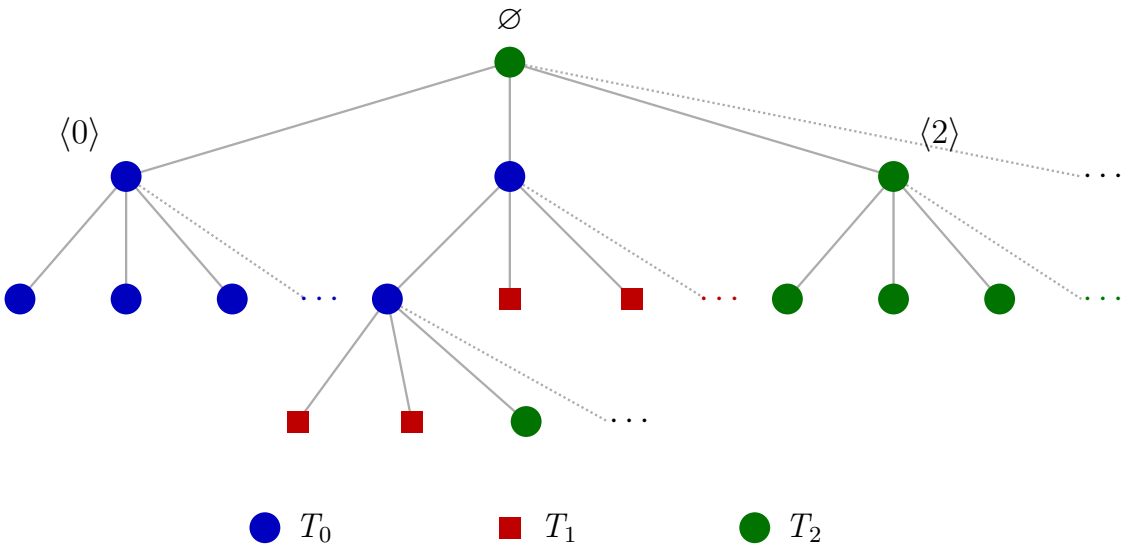
The remaining is $\Longleftarrow$.

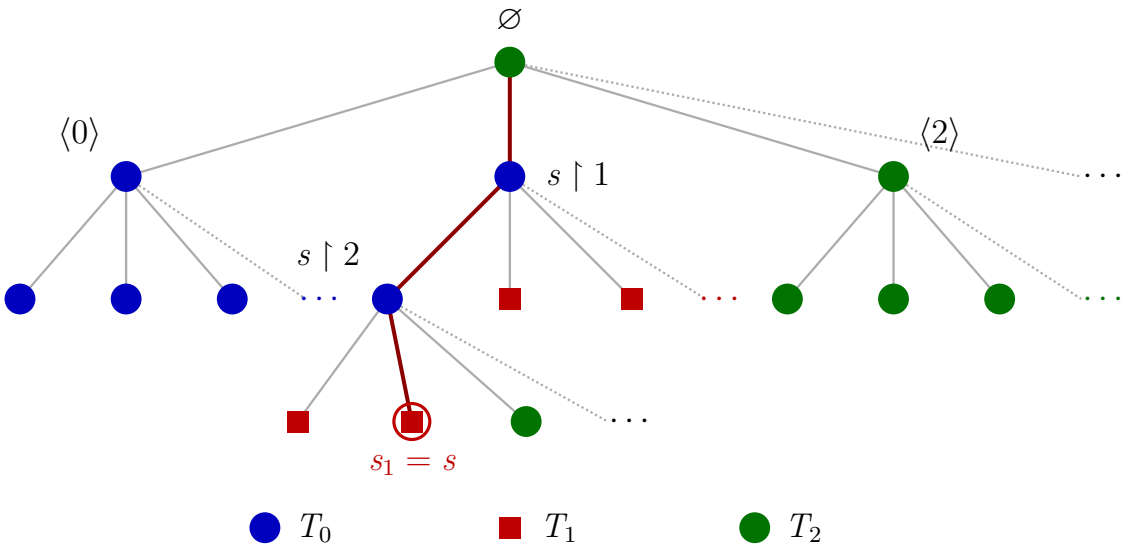
A technical lemma

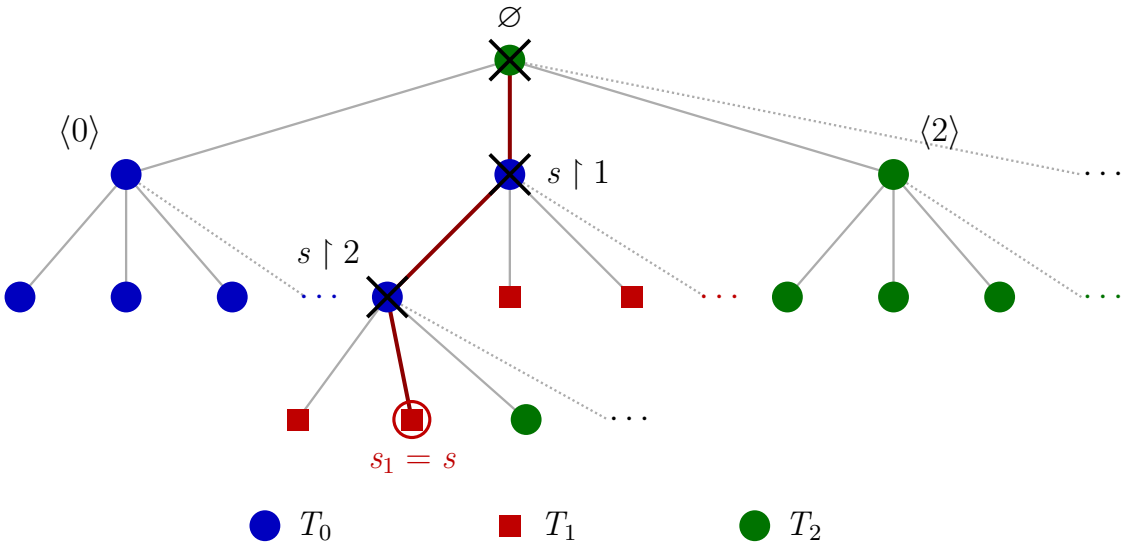
Lemma

If T is well-founded and $T_0, \dots, T_{k-1} \subseteq T$ are infinite, there are pairwise incomparable nodes $s_i \in T_i$, $i < k$.

The lemma can be proved using a figure.


∅
⟨0⟩
⟨2⟩
T_0
T_1
T_2


∅
⟨0⟩
⟨2⟩
s ⌈ 1
s ⌈ 2
s1 = s
T0
T1
T2


∅
⟨0⟩
⟨2⟩
s ↾ 1
s ↾ 2
s1 = s
T0
T1
T2

$\Longleftarrow$: A well-founded tree defeats every finite cover

Let T be well-founded. Fix $k \geq 1$ and infinite sets $c_0, \ldots, c_{k-1} \subseteq \omega$.

1. Apply the lemma to the infinite intersections $\psi[c_i] \cap T$ to choose pairwise incomparable nodes s_i. For every remaining index i, choose $s_i \in \psi[c_i] \setminus T$.
2. Put $p_i = \psi^{-1}(s_i) \in c_i$.
3. Define t long enough to contain all p_i with

$$t(j) = \begin{cases} 1, & j = p_i \text{ for some } i, \\ 0, & \text{otherwise}, \end{cases}$$

Then $E_t \cap T$ is an antichain, while $t(p_i) = 1 \neq 0$ for every $i < k$. Hence

$$t \in g(T) \setminus \bigcup_{i<k} S(c_i).$$

Thus $g(T) \notin \mathcal{S}pl$.

$\mathcal{S}pl$ is Σ^1_1-hard

It is easy to check g is continuous.
We have shown

$$T \in \mathrm{IF} \iff g(T) \in \mathcal{S}pl,$$

so $\mathrm{IF} \leq_W \mathcal{S}pl$. Together with $\mathcal{S}pl \in \Sigma^1_1$, this proves:

Theorem B

$\mathcal{S}pl$ is Σ^1_1-complete.

References


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Thank you!